5.1 Connected and Disconnected graphs

A graph is said to be **connected** if there exist at least one path between every pair of vertices otherwise graph is said to be **disconnected**. A null graph of more than one vertex is disconnected (Fig 3.12). Fig 3.9(a) is a connected graph where as Fig 3.13 are disconnected graphs.

A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a **component**. The graphs in fig 3.13 consists of two components.

![Null Graph of six vertices](image1)

Fig 3.12: Null Graph of six vertices

![Disconnected graph with two components](image2)

Fig 3.13: A disconnected graph with two components
5.2 Theorem 1:

A graph G is said to be disconnected if and only if its vertex V can be portioned into two nonempty, disjoint subsets v1 and v2 such that there exists no edge in G whose one end vertex is in subset v1 and the other in subset v2.

Proof:

Suppose that such a partitioning exists. Consider two arbitrary vertices a and b in G, such that a ∈ V1 and b ∈ V2. No path can be exists between vertices a and b; otherwise there would be at least one edge whose one end vertex would be in V1 and the other in V2. Hence if partition exists, G is not connected.

Conversely, let G be disconnected so there exists a vertex a in G and a vertex b in G such that there is no path between a and b in G.

Let V1 be the set of all vertices that are joined by path to a. Since G is disconnected V1 does not include all vertices of G.

Let V2 = V – V1 then V1 ∩ V2 = φ and V1 U V2 = V.

Thus the remaining vertices will form a (non-empty) set of V2 i.e. b ∈ V2. Thus no vertex in V1 is joined to any vertex in V2 by an edge.

5.3 Theorem 2:

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof:

Let G be a graph with all even degree of vertices except two vertices v1 and v2, which are odd degree. We know that number of vertices with odd degree in a graph is always an even. That is no graph can have an odd number of odd vertices. Therefore, in graph G, v1 and v2 must be belong to the same component and hence must have a path between them.
5.4 Theorem 3:

A simple graph (i.e., a graph without parallel edges or self-loops) with $n$ vertices and $k$ components can have at most $(n - k)(n - k + 1)/2$ edges.

Proof:

Let the number of vertices in each of the $k$-components of a graph be $n_1, n_2, \ldots, n_k$. Thus

$$n_1 + n_2 + \ldots + n_k = n \quad \text{or} \quad \sum_{i=1}^{k} n_i = n \quad 1 \leq i \leq k$$

Suppose a component with $n_i$ vertices, then maximum number of possible edges

$$= \frac{n_i(n_i - 1)}{2}, \text{ when it is complete.}$$

Hence, the maximum number of edges are

$$\frac{1}{2} \sum_{i=1}^{k} n_i(n_i - 1) = \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} \sum_{i=1}^{k} n_i$$

Now, we have

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

i.e.,

$$\left[ \sum_{i=1}^{k} (n_i - 1) \right]^2 = n^2 + k^2 - 2nk.$$

or

$$\sum_{i=1}^{k} (n_i^2 - 2n_i) + k + \text{non-negative cross terms} = n^2 + k^2 - 2nk$$

or

$$\sum_{i=1}^{k} n_i^2 \leq 2n - k + n^2 + k^2 - 2nk$$

or

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k - 1)(2n - k)$$

Putting this value in equation (i), we get

$$\frac{1}{2} \sum_{i=1}^{k} n_i(n_i - 1) \leq \frac{1}{2} \left[ n^2 - (k - 1)(2n - k) \right] - \frac{1}{2} n$$

$$= \frac{1}{2} [n^2 - 2nk - k^2 + n - k]$$

$$= \frac{1}{2} [n - k][n - k + 1]$$
5.5 Example:
Prove that a simple graph with $n$ vertices must be connected if it has more than $[(n - 1)(n - 2)]/2$ edges.

Solution:
Suppose $G$ is a simple graph with $n$ vertices
Choose $(n - 1)$ vertices such that $v_1, v_2, v_3, ....v_{n-1}$ of $G$.
We know that the maximum number of edges in a simple graph with $n$ vertices is $n(n-1)/2$.
So we $[(n - 1)(n - 2)]/2$ number of edges can be drawn for $(n-1)$ vertices. Thus if we have more than $[(n - 1)(n - 2)]/2$ edges than at least one edge should be drawn between the $n^{th}$ vertex i.e. $v_n$ to some vertex $v_i$.
Hence $G$ must be connected.

5.6 Example
Let $G$ be a disconnected graph with $n$ vertices where $n$ is even. If $G$ has two components each of which is complete, prove the $G$ has a minimum of $n(n - 1)/4$ edges.

Solution
Let $x$ be the number of vertices in one of the components than the other component has $(n - x)$ vertices. Since both components are complete graph.
We know that number of edges in a simple graph with $n$ vertices are $n(n-1)/2$.
Thus, the number of edges with $x$ and $(n - x)$ vertices are
$[x(x - 1)]/2$ and $[(n - x)(n - x - 1)]/2$
So, the total numbers of edges are
$E = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$
$= \frac{x^2 - x + n^2 - nx - n + x^2 + x}{2}$
$E = x^2 - nx + \frac{n}{2}(n - 1)$ ........................................... (i)
Differentiate w.r.t. $x$, we get
$E' = 2x - n$
Again differentiate w.r.t. $x$, we get
\[ E'' = 2 > 0 \]

Therefore, \( E \) is minimum when \( 2x - n > 0 \)

i.e. \[ x = \frac{n}{2} \]

For minimum value of \( n \), put the value of \( x \) in equation (i)

\[
E = \left( \frac{n}{2} \right)^2 - n \left( \frac{n}{2} \right) + \frac{n}{2}(n - 1)
\]

\[
= \frac{[n(n-1)]}{4}
\]